All Morphisms Are Equal, But Some Morphisms Are More Equal Than Others: An Introduction to Higher Category Theory

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Abstract

We introduce the concept of 2-categories and mention essential results. After examining the need for higher levels of abstraction, we give different constructions of infinity categories and show their equivalence. Finally, we will introduce the infinity category of spaces and investigate its role in the setting of higher category theory.

1. Set theoretic considerations

Throughout this report, we assume the existence of sufficiently many Grothendieck universes to avoid issues such as Russell's paradox.

Definition 1.1 (Inaccesible cardinal, Grothendieck)**.** We call a cardinal κ inaccessible if the collection of sets $\mathcal{V}_{<\kappa}$ of hereditary cardinality less than κ satisfies the ZFC axioms. Such a $\mathcal{V}_{<\kappa}$ is called a Grothendieck universe.

We should note that the existence of such cardinals is logically independent of ZFC and has to be taken as a separate axiom.

Remark 1.2. Note that with this definition, an inaccessible cardinal κ has to be greater than $\aleph_k$ for any k.

For the proper construction of categories, we will fix an increasing sequence

$$\kappa_0 < \kappa_1 < \kappa_2 < \cdots$$

of inaccesible cardinals. With this sequence, we define

Definition 1.3. A set is
(i) small if $S \in \mathcal{V}_{<\kappa_0}$
(ii) large if $S \in \mathcal{V}_{<\kappa_1}$
(iii) very large if $S \in \mathcal{V}_{<\kappa_2}$

As such, we note that the set of all small sets is large, and the set of all large sets is very large. The categories that we will work with will usually be large.

2. A review of 2-categories

2.1. **Motivation.** The essence of category theory is often summarized as highlighting morphisms between objects rather than objects themselves. Still, it may not always be enough to consider only these two pieces of data. In many contexts, how the morphisms relate to one another also carries significance. Moreover, many such morphism-relation data satisfy associativity and unitality laws, forming categories themselves.

Example 2.1. Consider the category Grp with groups as objects and group homomorphisms as morphisms. When studying groups, we are often concerned with homomorphisms only up to conjugacy (e.g., character theory, cohomology). For every pair of morphisms $f, f' \colon G \to H$ where G and H are objects of Grp, we call f and f' conjugate if there exists $h \in H$ such that

$$hf(g)h^{-1} = f'(g) \text{ for all } g \in G$$

This conjugation operation will turn out to "behave well" towards the category structure in the sense defined below.

Example 2.2 ($\mathcal{R}$el)**.** Another interesting category where we have relations between morphisms is the category $\mathcal{R}$el with objects small sets and morphisms binary relations between those sets. One can use the well-known idea of implications–which is just the inclusion of relations–to define a higher structure in this category.

2.2. **Basic definitions.** For brevity, we will present shortened versions of the 2-category definitions below, adapted from [7] and [6], respectively. An interested reader is encouraged to flesh these out.

Definition 2.3 (Strict 2-category)**.** A strict 2-category $\mathcal{C}$ consists of

- A collection of objects called 0-morphisms
- For every pair of objects $X, Y \in \mathcal{C}$, a category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose objects are called 1-morphisms and whose morphisms are called 2-morphisms. The composition in this category is called vertical composition and is shown by $*$.
- For every 0-morphism X, a distinguished 1-morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ called the identity 1-morphism at X.
- For every triple of 0-morphisms (X, Y, Z), a functor $\circ \colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ called horizontal composition such that
  - For every 1-morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$
  $$f \circ 1_X = 1_Y \circ f = f$$
  - For every 2-morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$
  $$\alpha \circ 1_{1_X} = 1_{1_Y} \circ \alpha = \alpha$$
  - For each triple of composable 1-morphisms h, g, f
  $$h \circ (g \circ f) = (h \circ g) \circ f$$
  - For each triple of composable 2-morphisms γ, β, α
  $$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$$

Unfortunately, strict associativity and unitality laws go against some natural constructions in higher category theory. For this reason, we seek to relax the definition of a strict 2-category.

Definition 2.4 (Weak 2-category)**.** A weak 2-category $\mathcal{C}$ is a 2-category where, instead of the horizontal composition, laws regarding associativity and unitality hold only up to natural isomorphisms called associators and unitors. For objects $W, X, Y, Z \in \mathcal{C}$, an associator is a natural isomorphism between functors

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \to \mathrm{Hom}_{\mathcal{C}}(W,Z).$$

giving the composition of three 1-morphisms, and for each pair of 0-morphisms x and y, the unitors are natural isomorphisms l and r identified below.

$$\begin{pmatrix} f & \mapsto & f \circ 1_x \\ \theta & \mapsto & \theta \circ 1_x \end{pmatrix} \overset{r}{\cong} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(x,y)} \overset{l}{\cong} \begin{pmatrix} f & \mapsto & 1_y \circ f \\ \theta & \mapsto & 1_y \circ \theta \end{pmatrix} : \mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}}(x,y)$$

Denoting horizontal composition $f \circ g$ as fg, for objects V, W, X, Y, Z and 1-morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(V,W)$, $g \in \mathrm{Hom}_{\mathcal{C}}(W,X)$, $h \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $k \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$, we have the following.

Unity Axiom: The middle unity diagram

$$\begin{tikzcd}
(g1_W)f \arrow[rr, "a"] \arrow[dr, "r_g * 1_f"'] & & g(1_W f) \arrow[dl, "1_g * l_f"] \\
& gf &
\end{tikzcd}$$

in $\mathrm{Hom}_{\mathcal{C}}(V,X)$ is commutative.

Pentagon Axiom: The diagram

$$\begin{tikzcd}
& (kh)(gf) \arrow[dr, "a_{k,h,gf}"] & \\
((kh)g)f \arrow[ur, "a_{kh,g,f}"] \arrow[d, "a_{k,h,g} * 1_f"'] & & k(h(gf)) \\
(k(hg))f \arrow[rr, "a_{k,hg,f}"'] & & k((hg)f) \arrow[u, "1_k * a_{h,g,f}"']
\end{tikzcd}$$

in $\mathrm{Hom}_{\mathcal{C}}(V,Z)$ is commutative where a denotes the associator.

This finishes the definition of a weak 2-category.

Example 2.1 and Example 2.2 are both strict 2-categories. The reader is advised to check that the composition laws do hold. For an example of a weak 2-category, we consider the following.

Example 2.5. Given a well-behaved topological space X, its fundamental 2-groupoid is the 2-groupoid whose

- objects are the points (elements) of X
- 1-morphisms are continuous paths $[0,1] \to X$
- 2-morphisms are homotopies between such paths, fixing their endpoints
- composition is given by concatenation of paths and homotopies

This definition gives a weak 2-category. Readers who need a refresher on homotopies are advised to refer to Definition 4.7.

A quick reader is correct in assuming that the definition of an n-category could be given recursively. Although, this unfortunately only works for *strict* n-categories.

To define a weak n-category using the established theory of $(n-1)$-categories is impossible since we require the associativity laws of $n-1$ morphisms to hold only up to n-isomorphism, which are not yet defined. Readers who are not comfortable with enriched categories are advised to refer to [6], Definition 1.3.1.

Definition 2.6 (Strict n-category)**.** A strict n-category is a category enriched over $\mathcal{C}\mathrm{at}_{n-1}$, where $\mathcal{C}\mathrm{at}_{n-1}$ is the $(n-1)$-category of strict $(n-2)$-categories with functors between them and equipped with the cartesian monoidal structure given by forming product categories.

Luckily, the notions of strict 2-categories and weak 2-categories are equivalent in some relevant sense. We formulate this "equivalence" in the following theorem. With these definitions, we recall an essential theorem from the theory of 2-categories. For the left to right direction of the proof, the reader is advised to refer to [6], Theorem 8.4.1.

Theorem 2.7. *Every strict* 2*-category is* 2*-equivalent to a weak* 2*-category, and every weak* 2*-category is biequivalent to a strict* 2*-category.*

For the relevant definitions of 2-equivalence and biequivalence, the reader is advised to refer to Definition 6.2.8 and Definition 6.2.10 in [6], respectively.

The situation is only worse for higher levels of abstraction. Although we will not give here a definition of strict or weak 3-categories, if one continues in the above manner and defines these structures, it becomes apparent that they are not as well-behaved. For example, not every weak 3-category is equivalent to a strict 3-category. Unfortunately, the smallest example of this fact involves quite a bit of homological algebra. We refer to section I.4.4 in [13] for a more detailed exposition.

Example 2.8. Simply connected 3-groupoids X have two homotopy groups $\pi_2(X), \pi_3(X)$, and are classified by this pair of homotopy groups together with a k-invariant, which is a cohomology class in $H^4(B^2\pi_2, \pi_3)$. We know that X is strictifiable if only if the k-invariant vanishes, which happens if and only if $X = B^2\pi_2 \times B^3\pi_3$, i.e., there are no nontrivial interactions between these homotopy groups. Then, the case of the fundamental 3-groupoid of S^2 corresponds to $\pi_2 = \pi_3 = \mathbb{Z}$ with k-invariant an element of $H^4(B^2\pi_2, \pi_3) \cong H^4(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}$ which as a cohomology operation $H^2(-, \mathbb{Z}) \to H^4(-, \mathbb{Z})$ is the cup square.

3. Motivation for ∞-categories

Let X be a topological space and $0 \le n \le \infty$. We can extract a weak n-category $\pi_{\le n}X$ as stated in [9].

- 0-morphisms of $\pi_{\le n}X$ are the points of X
- For $x, y \in \pi_{\le n}X$ a 1-morphism from x to y is a continuous path $[0,1] \to X$ starting at x and ending at y
- 2-morphisms are given by homotopies of paths
- 3-morphisms are given by homotopies of homotopies

$\vdots$

In some sort of limit, we hope to arrive at a theory of $(\infty, 0)$-categories, where every morphism is invertible up to homotopy. In what follows, we will try to generalize this notion and develop a theory of $(\infty, 1)$-categories, where every k-morphism for $k > 1$ is invertible up to homotopy. Unfortunately, it will be hard to formalize this idea.

4. Topological construction of ∞-categories

4.1. **A review of some topological concepts.** This section will review some topological concepts we will use to construct ∞-categories. Readers who have at least taken an introduction to topology course are encouraged to skip this section.

We start with the following useful notion of compactness, which will encode the idea of "containment" or "spreading out."

Definition 4.1 (Compact space)**.** A topological space X is compact if for every subset $K \subset X$ and every open subcover $\bigcup_\bullet \mathcal{O}_\bullet \supset K$ there exists a finite subcover $\bigcup_{i=0}^n \mathcal{O}_i \supset K$.

It is often important to understand how points in a topological space relate to each other. One of the major ways this can be achieved is to consider whether they are separable or to what extent. Many topological spaces one encounters in other fields satisfy strong separation axioms like being Hausdorff.

Definition 4.2 (Hausdorff space)**.** A topological space X is called Hausdorff if for every $x, y \in X$ such that $x \neq y$ there exists disjoint open sets $\mathcal{O}_x$ and $\mathcal{O}_y$ such that $x \in \mathcal{O}_x$ and $y \in \mathcal{O}_y$.

Unfortunately, the above definition will be too strong for the topological spaces we will be interested in. Instead, we will primarily consider spaces that are only weakly Hausdorff.

Definition 4.3 (Weakly Hausdorff)**.** A topological space X is called weakly Hausdorff if for every compact Hausdorff space K and every continuous map $f\colon K \to X$, the image $f(K)$ is closed in X.

Remark 4.4. We note that every Hausdorff space is necessarily weakly Hausdorff.

Although there are different equivalent definitions of compactly generated topological spaces, we give the one below because it is more intuitive in the context of category theory.

Definition 4.5 (Compactly generated)**.** A topological space X is compactly generated if

$$\text{A subset } \mathcal{O} \subset X \text{ is open in } X \iff \begin{array}{l}\text{For every compact Hausdorff topological}\\ \text{space K and every continuous function}\\ f\colon K \to X \text{ the inverse image } f^{-1}(\mathcal{O}) \text{ is open.}\end{array}$$

Some topological spaces can be "built up" by gluing certain topological objects. A particularly well-studied type of such space is a CW complex, which we define below.

Definition 4.6 (CW complex)**.** Consider a sequence of spaces $X^0 \subseteq X^1 \subseteq X^2 \subseteq \ldots$, where each X^n is the space formed by attaching n-dimensional cells to X^{n-1} via continuous maps:

(i) X^0 is just a discrete set.
(ii) Form an n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha \colon S^{n-1} \to X^{n-1}$. Equivalently, take X^n to be the quotient space of the coproduct $X^{n-1} \coprod_\alpha D^n_\alpha$ in the category Top where D^n_α is a collection of n-disks under the identification $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$ where e^n_α is an open disk. This process is encoded in the following pushout diagram:

$$\begin{tikzcd}
\coprod_\alpha S^{n-1}_\alpha \arrow[r] \arrow[d] & X^{n-1} \arrow[d] \\
\coprod_\alpha D^n_\alpha \arrow[r] & X^n
\end{tikzcd}$$

Then, the colimit of the sequence in Top is called a CW complex, with the final topology induced by the maps from each X^n to X.

We now focus on continuous maps of topological spaces rather than spaces themselves.

Definition 4.7 (Homotopy)**.** Given topological spaces X, Y and continous maps $f, g \colon X \to Y$, a homotopy H between f and g is a continous map

$$H \colon X \times [0,1] \to Y$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Remark 4.8. Homotopies can be thought of as continuous deformations of continuous functions. It should be noted that since homotopies themselves are continuous maps, homotopies of homotopies are well-defined.

4.2. **Homotopy Hypothesis.** With the example in section 3., it is not unreasonable to wish an ∞-category to be a collection of objects and an infinity groupoid $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for each $X, Y \in \mathcal{C}$ which we will identify with topological spaces. Taking $(\infty,0)$-categories to be the same thing as homotopy types is called the "Homotopy Hypothesis."

4.3. **Construction of ∞-categories.**

Definition 4.9 (Topological categories)**.** A topological category is a category enriched over category $\mathcal{CG}$ of compactly generated and weakly Hausdorff topological spaces.

Remark 4.10. We wish to construct ∞-categories from $\mathcal{CG}$-enriched categories rather than Top-enriched categories. This is mainly due to cartesian closure. $\mathcal{CG}$ is cartesian closed while Top is not. This means that a functor $- \times T$ has a right adjoint $\mathrm{Hom}_{\mathcal{C}}(Y, -)$. Thus, for any objects $X, Y \in \mathcal{C}$, we can choose $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ to be the corresponding hom-object, which is just the space of continuous functions with the compactly generated topology induced by compact-open topology.

Definition 4.11 (∞-category)**.** We define an ∞-category to be a topological category.

Although this is the most transparent and effortless definition to give of an ∞-category, it comes with a major problem: 1-morphisms in a topological category are strictly associative. So, to stay in the world of topological categories, we need to "straighten" our 1-morphisms constantly, using a "Quillen equivalence," which we will touch on later. This is highly impractical and leads to a difficult theory of higher categories.

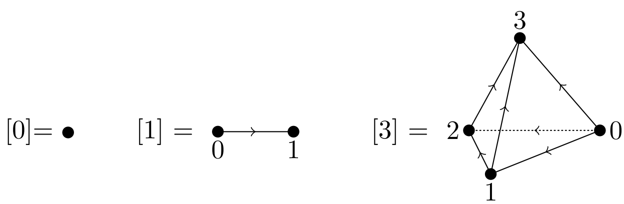
5. Simplicial construction ∞-categories

This chapter aims to define ∞-categories in a "purely combinatorial" manner. Just as we can study topological spaces more easily using a combinatorial approach via simplices, we can apply the same approach to ∞-categories. We will first try to build a categorical foundation for simplical concepts and then construct ∞-categories using these concepts.

Definition 5.1 (Simplex category)**.** We define a category Δ, called the category of simplices or the simplex category, consisting of the following data:

- Objects linearly ordered sets $[n] := \{0, 1, \dots, n\}$ for every $n \geq 0$.
- Morphisms weakly monotone maps, i.e., $f\colon [m] \to [n]$ such that $a \leq b$ implies $f(a) \leq f(b)$.

Remark 5.2. The objects of Δ can be drawn as simplices with ordered vertices. For example, we have



With this definition, one can easily show that there are only $\binom{n+m-1}{n}$ weakly monotone maps from $[n]$ to $[m]$. However, we need not always consider all of these maps when studying Δ since they can be generated from some special types of weakly monotone maps.

Definition 5.3 (Coface, codegeneracy maps)**.** In the category Δ, we define special morphisms called coface maps

$$\begin{aligned} \delta_i^n \colon [n-1] &\longrightarrow [n] \\ k &\longmapsto \begin{cases} k, & \text{if } k < i \\ k+1, & \text{if } k \geq i \end{cases} \end{aligned}$$

for $n \geq 1$ and $0 \leq i \leq n$. We also define codegeneracy maps

$$\sigma_i^n \colon [n+1] \longrightarrow [n]$$
$$k \longmapsto \begin{cases} k, & \text{if } k \leq i \\ k-1, & \text{if } k > i \end{cases}$$

for $n \geq 0$ and $0 \leq i \leq n$.

Proposition 5.4. In Δ, any morphism $f\colon [n] \to [m]$ has a unique representation

$$f = \delta_{i_k}^n \delta_{i_{k-1}}^n \cdots \delta_{i_1}^n \sigma_{j_1}^n \sigma_{j_2}^n \cdots \sigma_{j_h}^n$$

where $n, k \geq 0$ satisfy $k - h = m - n$ and the indices are such that

$$m \geq i_k > i_{k-1} > \cdots > i_1 \geq 0, \quad 0 \leq j_1 < j_2 < \cdots < j_h < n$$

Proof. A monotonically nondecreasing function f is completely determined by its image and by $j \in [n]$ for which f does not increase, i.e., $j \in [n]$ such that $f(j) = f(j+1)$.

Let $i_1, i_2, \ldots, i_k$, in increasing order, be the elements of $[m]$ which are not in the image of f, and $j_1, j_2, \ldots, j_h$ be the elements of $[n]$ for which f does not increase, also in increasing order. The result follows. $\square$

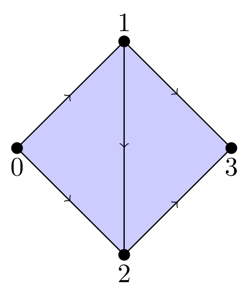
Before continuing with the categorification, we recall what a finite simplicial complex is.

Definition 5.5 (Finite simplicial complex)**.** A finite simplicial complex K consists of a set V_K of vertices and a set $\mathrm{Sim}(K) \subset \mathcal{P}(V_K) \setminus \varnothing$ of simplices such that

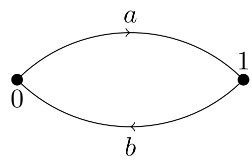
- If $\varnothing \neq U \subset V \in \mathrm{Sim}(K)$ then $U \in \mathrm{Sim}(K)$
- For all $S \in V_K$, the singleton $\{S\}$ is in $\mathrm{Sim}(K)$.

Remark 5.6. We may visualize a finite simplicial complex as the join of its simplices.

Example 5.7. Let K be a complex such that $V_K = \{0, 1, 2, 3\}$ and let $\mathrm{Sim}(V_K)$ be a superset of $\{\{0, 1, 2\}, \{1, 2, 3\}\}$. Then we may draw K as



We now explore why finite simplicial complexes are inadequate for building our categories, as stated in [14]. Consider the following diagram



Note that this is not a finite simplicial complex since in a finite simplicial complex, the simplices are completely determined by their vertices, and hence, there can only be a single 1-simplex between any two vertices. One remedy to this problem would be to consider sets X_n of n-simplices for each $n \geq 0$ instead of $\mathrm{Sim}(K)$. In our case, this new definition boils down to defining sets

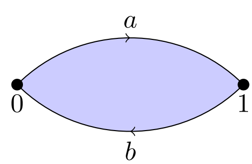
$$\begin{aligned} X_0 &= \{0, 1\} \\ X_1 &= \{a, b\} \\ X_k &= \varnothing \text{ for } k \geq 2 \end{aligned}$$

Unfortunately, this definition forgets the face information for each vertex. However, we can encode this as additional data from the face maps we defined earlier. Namely

$$\begin{aligned} d_0 \colon X_1 &\to X_0 \\ a &\mapsto 1 \\ b &\mapsto 0 \end{aligned} \qquad\qquad \begin{aligned} d_1 \colon X_1 &\to X_0 \\ a &\mapsto 0 \\ b &\mapsto 1 \end{aligned}$$

where d_i deletes the i-th vertex of each 1-simplex and matches it to the remaining vertex.

Now we consider a slightly harder problem. Suppose that the inside of the above structure is filled by a 2-simplex.



Again, we face the inflexibility of finite simplicial complexes, but we can solve the problem like the one mentioned above. Using degeneracy maps, we can define a 1-simplex that collapses to 0 using degeneracy maps. That is, we define $s_i \colon X_n \to X_{n+1}$ such that $s_i(\sigma) \in X_{n+1}$ is degenerate. Such a construction will lead us to the definition of simplicial sets.

Remark 5.8. Note that for the above discussion to make sense, we must lose all hope of finiteness, for if we have even a single 0-simplex x in our complex, then we will have

$$X_0 = \{x\}, \quad X_1 = \{s_0^0(x)\}, \quad X_2 = \{s_0^1(s_0^0(x))\}, \ldots$$

Now, we will state some identities to generalize the properties of coface and cogeneracy maps, and then use them to uniquely define functors out of Δ.

Definition 5.9 (Cosimplicial identities)**.**

- $\delta_j \delta_i = \delta_i \delta_{j-1}$ if $i < j$
- $\sigma_j \delta_i = \delta_i \sigma_{j-1}$ if $i < j$
- $\sigma_j \delta_j = \mathrm{id} = \sigma_j \delta_{j+1}$
- $\sigma_j \delta_i = \delta_{i-1} \sigma_j$ if $i > j+1$
- $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$ if $i \leq j$

Remark 5.10. It can be routinely checked that the coface and codegeneracy maps satisfy he cosimplicial identities.

Proposition 5.11. Let $\mathcal{C}$ be a category. The following data uniquely defines a functor $F\colon \Delta \to \mathcal{C}$.

- Objects $F([n])$ for all $n \geq 0$.
- For every coface δ_i^n, a morphism
$$F(\delta_i^n)\colon F([n-1]) \to F([n])$$
and for every codegeneracy σ_i^n, a morphism
$$F(\sigma_i^n)\colon F([n+1]) \to F([n])$$
both of which satisfy the cosimplicial identities.

Proof. This proposition follows from Proposition 5.4 after checking unitality and associativity, which is a result of the cosimplicial identities. □

Definition 5.12 (Simplicial identities)**.**

- $d_i d_j = d_{j-1} d_i$ if $i < j$
- $d_i s_j = s_{j-1} d_i$ if $i < j$
- $d_j s_j = \mathrm{id} = d_{j+1} s_j$
- $d_i s_j = s_j d_{i-1}$ if $i > j+1$
- $s_i s_j = s_{j+1} s_i$ if $i \leq j$

Remark 5.13. Proposition 5.11 has a dual for $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ where the morphisms must satisfy simplicial identites instead.

With the above setup we are now ready to categorify the notion of a simplicial complex.

Definition 5.14 (Simplicial set)**.** A functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ with

- Sets $X_n := F([n])$ called n-simplices for every $n \geq 0$.
- Maps
$$d_i^n\colon X_n \to X_{n-1}$$
and
$$s_i^n\colon X_n \to X_{n+1}$$
called the i-th face and degeneracy maps, respectively, satisfying the simplicial identities

is called a simplicial set.

Of course, as usual for category theory, the set of simplicial sets can be organized into a category.

Definition 5.15 (Set$_\Delta$)**.** We define the category Set$_\Delta$ to be the category of presheaves on Δ, i.e., $\mathrm{Set}_\Delta := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. A morphism in this category is just a natural transformation $X \Rightarrow Y$, which amounts to arrows $X_n \to Y_n$ commuting with the face and degeneracy maps.

Now, we will define what it means for a simplex of a simplicial set to be degenerate as we did earlier in the context of simplicial complexes.

Definition 5.16 (Degenerate simplex)**.** Let X be a simplicial set. A simplex $\sigma \in X_n$ is called degenerate if it is in the image of a degeneracy map.

With these definitions, we can describe some special simplicial sets that we will use for constructing ∞-categories.

Example 5.17. Let J be a linearly ordered set. We denote by Δ^J the representable functor $\mathrm{Hom}(-, J)\colon \Delta \to \mathrm{Set}_\Delta$.

Example 5.18 (Standard n-simplex)**.** When $J = [n]$ in the above example, we denote $\Delta^{[n]} = \Delta^n$ and call it the standard (combinatorial) n-simplex.

The above definition suggests the use of Yoneda's lemma.

Remark 5.19. For any simplicial set X we have the natural identification $X_k \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^k, X)$ by the Yoneda lemma. Then, with this identification, we can picture the standard n-simplex as ordered simplices.

Now, we wish to define a functor from Δ to $\mathcal{CG}$ to make precise the pictures we have been drawing, as presented in [14].

Definition 5.20 (Geometrical n-simplex)**.** For $n \geq 0$ the geometrical n-simplex $|\Delta^n|$ is given by

$$|\Delta^n| = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; t_i \geq 0 \text{ and } \sum_{n=0}^{n} t_i = 1 \right\}$$

Definition 5.21 (Geometric realization functor)**.** We define a functor

$$\begin{aligned} |-|\colon \Delta &\to \mathcal{CG} \\ [n] &\mapsto |\Delta^n| \end{aligned}$$

with the following data:

- For every object $[n]$ a geometric simplex $|\Delta^n|$
- For every coface δ_i^n a map

$$\begin{aligned} |\delta_i^n|\colon |\Delta^{n-1}| &\to |\Delta^n| \\ (t_0, \ldots, t_{n-1}) &\mapsto (t_0, \ldots, \underbrace{0}_{i-\mathrm{th}}, \ldots, t_{n-1}) \end{aligned}$$

- For every codegeneracy σ_i^n a map

$$|\sigma_i^n|\colon |\Delta^{n+1}| \to |\Delta^n|$$
$$(t_0, t_1, \dots, t_n) \mapsto (t_0, \dots, t_{i-1}, t_i + t_{i+1}, \dots, t_{n+1})$$

Remark 5.22. More generally, for a morphism $\phi\colon \Delta^n \to \Delta^m$ we get a morphism

$$|\phi|\colon |\Delta^n| \to |\Delta^m|$$
$$(t_0, \dots, t_n) \mapsto (a_0, \dots, a_m)$$

where $a_i := \sum_{i \in \phi^{-1}(j)} t_i$

The functor defined above extends to Set_Δ naturally by taking its left Kan extension along the Yoneda embedding $\Delta \to \mathrm{Set}_\Delta$. The adjunction this functor provides will be key to understanding the relationship between simplicial sets and topological spaces.

Definition 5.23 (Sing X)**.** The geometric realization functor

$$| - |\colon \mathrm{Set}_\Delta \to \mathcal{CG}$$

is the left Kan extension of $| - |\colon \Delta \to \mathcal{CG}$ along the Yoneda embedding $\Delta \to \mathrm{Set}_\Delta$. It has a right adjoint called the singular simplicial set functor:

$$\mathrm{Sing}\colon \mathcal{CG} \to \mathrm{Set}_\Delta$$
$$X \mapsto \mathrm{Hom}_{\mathcal{CG}}(|\Delta^{(-)}|, X)$$

That is, an n-simplex in Sing X is a continuous map $|\Delta^n| \to X$.

Remark 5.24. We may realize this definition more geometrically by using coproducts and coequalizers. First note that $|X| = \mathrm{colim}_{\Delta^k \to X} \Delta^k$. By basic 1-category theory, we may write this colimit as the coequalizer of

$$\coprod_{\Delta^k \to \Delta^n \to X} |\Delta^k| \underset{f_*}{\overset{\mathrm{id}}{\rightrightarrows}} \coprod_{\Delta^m \to X} |\Delta^m|$$

But this is just a quotient of the space

$$\coprod_{n \geq 0} X_n \times |\Delta^n|$$

by relation

$$(f^*(\sigma), x) \sim (\sigma, f_*(x))$$

where $f\colon [n] \to [m]$, $\sigma \in X_m$, and $x \in |\Delta^n|$.

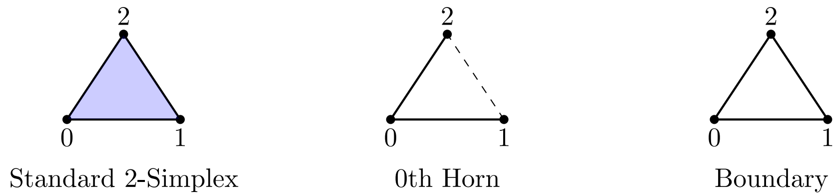
This effectively means that the geometric realization of a simplicial set is attained by gluing together the geometric realizations of its n-simplices in X_n according to face and degeneracy maps.

After this little detour, we continue our discussion of important simplicial sets.

Definition 5.25 (Boundary)**.** Fix $n \geq 0$. Then, the smallest simplicial set containing all faces of Δ^n is called the boundary of Δ^n, and it is denoted by $\partial\Delta^n$.

Definition 5.26 (Horn)**.** Fix $n \geq 0$ and $i \in [n]$. Then the set of all order-preserving morphisms $p\colon [m] \to [n]$ such that $p([m]) \cup \{i\} \neq [n]$ is called the i-th horn of Δ^n, and it is denoted by Λ^n_i.

There is a visual interpretation of these simplicial sets in the vein of the standard n-simplex. The boundary of Δ^n corresponds geometrically to the n-simplex with ordered vertices without the greatest simplex, i.e., the interior. The horn is the standard n-simplex with the interior and the i-th face scooped out.



Definition 5.27 (Inner horn)**.** If $0 < i < n$, then Λ^n_i is called an inner horn.

A fundamental tool in the construction of ∞-categories used to bridge simplicial sets to ordinary categories is the nerve functor. In the next few definitions, we aim to build this functor in a natural and intuitively appealing manner, which will mirror the discussion in [14].

Definition 5.28 (Functor τ and category τ^n)**.** We define the functor

$$\tau\colon \Delta \to \mathcal{C}\text{at}$$
$$[n] \mapsto \tau^n$$

where τ^n is the category given by object $1, 2, \ldots, n$ and

$$\text{Hom}_{\tau^n}(i,j) = \begin{cases} *, & i \leq j \\ \varnothing, & i > j \end{cases}$$

One can visualize the category τ^n as a diagram

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

Definition 5.29 (Fundamental category functor)**.** Consider the left Kan extension of τ along

$$\mathcal{Y}\colon \Delta \to \text{Set}_\Delta$$
$$[n] \mapsto \Delta^n$$

given by

$$\text{Lan}_{\mathcal{Y}}\tau\colon \text{Set}_\Delta \to \mathcal{C}\text{at}$$
$$X \mapsto \text{colim}_{\Delta^k \to X} \Delta^k$$

This map is called the fundamental category functor.

Proposition 5.30 (Nerve functor)**.** The fundamental category functor has a right adjoint called the nerve functor.

$$\begin{aligned} \mathrm{N}\colon \mathcal{C}\mathrm{at} &\to \mathrm{Set}_\Delta \\ \mathcal{C} &\mapsto \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\tau^{(-)}, \mathcal{C}) \end{aligned}$$

Proof.

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Set}_\Delta}(X, \mathrm{N}(\mathcal{C})) &\cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{colim}_{\Delta^k \to X}\Delta^k, \mathrm{N}(\mathcal{C})) \\ &\cong \lim_{\Delta^k \to X}\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^k, \mathrm{N}(\mathcal{C})) \\ &\cong \lim_{\Delta^k \to X}\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\tau^k, \mathcal{C}) \\ &\cong \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\lim_{\Delta^k \to X}\tau^k, \mathcal{C}) \\ &\cong \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathrm{Lan}_{\mathcal{Y}}\tau(X), \mathcal{C}) \end{aligned}$$

□

Proposition 5.31. The functor $N\colon \mathcal{C}\mathrm{at} \to \mathrm{Set}_\Delta$ is fully faithful.

Proof. This follows from the fact that we have a canonical isomorphism

$$\mathrm{colim}_{\tau^k \to \mathcal{C}}\tau^k \to \mathcal{C}$$

Thus the counit $\epsilon\colon \mathrm{Lan}_{\mathcal{Y}}\tau \circ \mathrm{N} \Rightarrow \mathrm{id}_{\mathcal{C}\mathrm{at}}$ is an isomorphism which implies that N is fully faithful. □

Definition 5.32 (Nerve)**.** For category $\mathcal{C}$, its nerve $\mathrm{N}(\mathcal{C})$ consists of the following data.

- Functors $\tau^0 \to \mathcal{C}$ as 0-simplices (i.e. the objects of $\mathcal{C}$)
- Functors $\tau^1 \to \mathcal{C}$ as 1-simplices which correspond to diagrams $x_o \xrightarrow{f} x_1$ in $\mathcal{C}$.
- Functors $\tau^n \to \mathcal{C}$ as n-simplices which correspond to diagrams of length n in $\mathcal{C}$

$$x_o \xrightarrow{f_{01}} x_1 \xrightarrow{f_{12}} x_2 \xrightarrow{f_{23}} \cdots \xrightarrow{f_{(n-1)n}} x_n$$

Remark 5.33. With the above definition, we see that $\mathcal{C}$ may be recovered up to isomorphism from its nerve.

- The objects of $\mathcal{C}$ are given by the 0-simplices of the nerve.
- A morphism from an object c_0 to an object c_1 is given by 1-simplex ϕ of the nerve such that $d_0(\phi) = c_1$ and $d_1(\phi) = c_0$.
- For an object c of $\mathcal{C}$, the identity morphism id_c is given by the degenerate simplex $s_0(c)$.
- Finally, given a diagram $c_0 \xrightarrow{\phi} c_1 \xrightarrow{\psi} c_2$ the edge of $\mathrm{N}(\mathcal{C})$ corresponding to $\psi \circ \phi$ is characterized uniquely by the fact that there is a unique $\sigma \in \mathrm{N}(\mathcal{C})_2$ with $d_2(\sigma) = \phi$, $d_0(\sigma) = \psi$, $d_1(\sigma) = \psi \circ \phi$.

The composition and unitality laws can be checked easily from these definitions.

Now, we examine some properties of nerves, which will lead us to the structures upon which we will define ∞-categories.

Definition 5.34 (Horn fillers)**.** Let X be a simplicial set and Λ_i^n be the i-th horn of the standard n-simplex with a map $f\colon \Lambda_i^n \to X$. By a filler for Λ_i^n we mean a dotted arrow g such that the following diagram commutes:

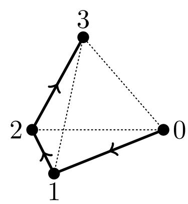
$$\begin{tikzcd}
\Lambda_i^n \arrow[r, "f"] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

As such, we say that X admits a filler for the i-th horn.

Definition 5.35 (Spine)**.** The spine of an n-simplex Δ^n is the pushout

$$\mathrm{Sp}(\Delta^n) = \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subset \Delta^n$$

Remark 5.36. Graphically, we may view the spine of a 3-simplex as



Lemma 5.37. Let $\mathcal{C}$ be a category. The restriction along the spine inclusion $\iota_n\colon \mathrm{Sp}(\Delta^n) \to \Delta^n$ gives

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, N(\mathcal{C})) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sp}(\Delta^n), N(\mathcal{C}))$$

Now, we are ready to prove an important result that determines nerves of small categories up to isomorphism.

Proposition 5.38. Let X be a simplicial set. Then the following conditions are equivalent:

(1) There exists a small category $\mathcal{C}$ and an isomorphism $X \cong N(\mathcal{C})$.
(2) X admits unique inner horn fillers.

Proof. Suppose that there exists a small category $\mathcal{C}$ such that $X \cong \mathrm{N}(\mathcal{C})$. Note that for any $n \geq 2$ and $0 < i < n$ the spine inclusion $\mathrm{Sp}(\Delta^n) \to \Delta^n$ factors through the horn inclusion $\Lambda_i^n \to \Delta^n$.

$$\mathrm{Sp}(\Delta^n) \hookrightarrow \Lambda_i^n \hookrightarrow \Delta^n$$

Then by Lemma 5.37 we have

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{N}(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda_i^n, \mathrm{N}(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sp}(\Delta^n), \mathrm{N}(\mathcal{C}))$$

Then, if the latter map is injective, we are done. By again Lemma 5.37, for $\phi\colon \Lambda^n_i \to \mathrm{N}(\mathcal{C})$ the image of each $(n-1)$-simplex in Λ^n_i is uniquely determined by its spine.

The $(2) \implies (1)$ part of the proof follows from Remark 5.33. □

Corollary 5.39. The nerve functor descends to an equivalence of categories between $\mathcal{C}$at and the full subcategory of Set_Δ on the simplicial sets, which admit unique fillers for all inner horns.

Corollary 5.40. Let $X \in \mathrm{Set}_\Delta$. There exists a groupoid $\mathcal{G}$ and isomorphism $X \cong N(\mathcal{G})$ if and only if X admits unique fillers for all horns.

Note that a 2-simplex $\sigma\colon \Delta^2 \to K$ is just a diagram

$$\begin{tikzcd} & Y \arrow[dr, "\psi"] & \\ X \arrow[ur, "\phi"] \arrow[rr, "\theta"'] & & Z \end{tikzcd}$$

together with an identification between θ and $\psi \circ \phi$. In general, even if 1-morphisms can be identified with 1-simplices, we cannot hope for the existence–let alone uniqueness–of a 2-simplex giving the above identification, unlike in the case of nerves of categories.

The existence of σ has to be given as an additional datum, which we may formulate using the horn extension property: for any $\Lambda^2_1 \to K$, there exists a dotted arrow $\Delta^2 \to K$ such that

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & K \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

commutes. The existence of such an extension is already satisfied, but it turns out that the uniqueness is not actually what we seek, for it goes against the homotopy hypothesis.

Consider the fundamental groupoid $\pi_1(X)$ of some topological space X. Note that there are many ways to take composites of paths in $\pi_1(X)$, which are the 1-morphisms. The following examples define the composition of two paths: p and q.

$$r(t) = \begin{cases} p(2t), & 0 \leq t \leq 1/2 \\ q(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

$$r' = \begin{cases} p(3t), & 0 \leq t \leq 1/3 \\ q(\frac{3t-1}{2}), & 1/3 \leq t \leq 1 \end{cases}$$

Note that neither of these maps is better than the other, as they are homotopic to each other. With the homotopy hypothesis, we adapt this ambiguity as a principle, so nerves are not exactly the building blocks we seek for ∞-categories.

Our next attempt will be to remove the uniqueness and require all horn fillers to exist.

Definition 5.41 (Kan complex)**.** A simplicial set X is called a Kan complex if it has fillers for all horns.

The following proposition is easily attained from the definitions.

Proposition 5.42. Let X be a topological space. Then Sing X is a Kan complex.

Remark 5.43. Any Kan complex behaves like a topological space. There are combinatorial recipes for extracting homotopy groups from K, which are isomorphic to the homotopy groups of the topological space $|K|$.

Unfortunately, a Kan complex is also not what we seek to define as an $(\infty, 1)$-category. The extension property for outer horns provides us with 1-morphisms, which are invertible up to homotopy, which would correspond to an ∞-groupoid instead.

5.1. **Construction of ∞-categories.** We start by noting that asking an ∞-category to be a collection of objects and an infinity groupoid $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for each $X, Y \in \mathcal{C}$ precisely corresponds to the common weakening of the Kan complex and nerve axioms.

Definition 5.44 (∞-category)**.** A simplicial set X, which has fillers for all inner horns, is called an ∞-category.

Remark 5.45. Note that we do not require these horn fillers to be unique.

Remark 5.46. Given an ∞-category $\mathcal{C}$ we can view the elements of the set $\mathcal{C}_0$ as objects (0-morphisms) and $\mathcal{C}_1$ as morphisms between those objects. The two face maps $\mathcal{C}_1 \rightrightarrows \mathcal{C}_0$ define these morphisms' source and target similarly to the constructions above. For $k > 1$, the elements of $\mathcal{C}_k$ are the higher morphisms.

Remark 5.47. Note that, for example, the map $\Lambda^2_1 \to \mathcal{C}$ corresponds to a diagram

$$\begin{tikzcd} & c_1 \arrow[dr, "f_{12}"] & \\ c_0 \arrow[ur, "f_{01}"] & & c_2 \end{tikzcd}$$

and when we say that $\mathcal{C}$ admits inner horns, we mean the existence of a dotted arrow f_{02} such that the diagram

$$\begin{tikzcd} & c_1 \arrow[dr, "f_{12}"] & \\ c_0 \arrow[ur, "f_{01}"] \arrow[rr, dashed, "f_{02}"'] & & c_2 \end{tikzcd}$$

commutes. Choosing such a map is not equivalent to choosing an extension $\Delta^2 \to \mathcal{C}$ but instead an extension $\partial\Delta^2 \to \mathcal{C}$. For the former, we also have to choose a homotopy between f_{01}, f_{12} and f_{02}. In the nerve of a category, such an extension is unique, which is why we can recover a category with a well-defined composition law from it. On the other hand, for an ∞-category, we do not have uniqueness, which prevents us from having well-defined composition for morphisms.

Example 5.48 (Every Kan complex is an infinity category)**.** By definition, a Kan complex is a simplicial set with fillers for every horn, particularly for inner horns. Thus, it is also an ∞-category.

Example 5.49 (Every nerve is an infinity category)**.** A nerve has unique fillers for every inner horn by Proposition 5.38. Then, it is directly an ∞-category. Moreover, by Remark 5.33, we may confuse a nerve with its category and assert that all 1-categories are also ∞-categories. Without this identification, we may still refer to a 1-category as an ∞-category simply by using its associative law of morphisms to attain inner horn fillers.

6. Model categories and equivalence of constructions

This section aims to show that our two different constructions for ∞-categories are equivalent in some relevant sense. First, we will have to build the required framework and define what it means for two constructions to be the same. We start with introducing model categories below, as stated in [5].

Definition 6.1 ($\operatorname{Map}\mathcal{C}$)**.** Given a category $\mathcal{C}$, we can form the category $\operatorname{Map}\mathcal{C}$ whose objects are morphisms of $\mathcal{C}$ and whose morphisms are commutative squares.

Definition 6.2 (Retract, functorial factorization)**.** Suppose $\mathcal{C}$ is a category.

(1) A map f in $\mathcal{C}$ is a retract of a map $g \in \mathcal{C}$ if f is a retract of g as objects of $\operatorname{Map}\mathcal{C}$. That is, f is a retract of g if and only if there is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

where the horizontal composites are identities.

(2) A functorial factorization is an ordered pair (α, β) of functors $\operatorname{Map}\mathcal{C} \to \operatorname{Map}\mathcal{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in \operatorname{Map}\mathcal{C}$. In particular, the domain of $\alpha(f)$ is the domain of f, the codomain of $\alpha(f)$ is the domain of $\beta(f)$, and the codomain of $\beta(f)$ is the codomain of f.

Definition 6.3 (Lifting properties)**.** Suppose $i : A \to B$ and $p : X \to Y$ are maps in a category $\mathcal{C}$. Then i has the left lifting property with respect to p and p has the right lifting property with respect to i if, for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there is a lift $h : B \to X$ such that $hi = f$ and $ph = g$.

Definition 6.4 (Model structure)**.** A model structure on a category $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following properties:

(1) (2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.
(2) (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g, and g is a weak equivalence, cofibration, or fibration, then so is f.
(3) (Lifting) Define a map to be a trivial cofibration if it is both a cofibration and a weak equivalence. Similarly, define a map to be a trivial fibration if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.
(4) (Factorization) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

Now that we have some structure to encode homotopical information, we can define adjunctions to relate different homotopical structures.

Definition 6.5 (Quillen adjunction)**.** Let

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$$

be an adjunction. It is called a Quillen adjunction if G preserves fibrations and F preserves cofibrations.

Lemma 6.6. (Ken Brown's Lemma) Let $F \dashv G$ be a Quillen adjunction. Then G preserves weak equivalences of fibration objects, and F preserves weak equivalences of cofibration objects.

Now, we define a new notion of equivalence, as stated by Quillen, which will help us show the "sameness" of the homotopical information of our different constructions. This is the "correct" kind of equivalence because we only care about the data in our constructions up to homotopy.

Definition 6.7 (Homotopy category)**.** Suppose $\mathcal{C}$ is a category with a subcategory of weak equivalences $\mathcal{W}$. Define the homotopy category $\mathrm{Ho}\mathcal{C}$ as follows. Form the free category $F(\mathcal{C}, \mathcal{W}^{-1})$ on the arrows of $\mathcal{C}$ and the reversals of the arrows of $\mathcal{W}$. An object of $F(\mathcal{C}, \mathcal{W}^{-1})$ is an object of $\mathcal{C}$, and a morphism is a finite string of composable arrows $(f_1, f_2, \ldots, f_n)$ where f_i is either an arrow of $\mathcal{C}$ or the reversal w_i^{-1} of an arrow w_i of $\mathcal{W}$. The empty string at a particular object is the identity at that object, and composition is defined by the concatenation of strings. Now, define $\mathrm{Ho}\mathcal{C}$ to be the quotient category of $F(\mathcal{C}, \mathcal{W}^{-1})$ by the relations $1_A = (1_A)$ for all objects A, $(f, g) = (g \circ f)$ for all composable arrows f, g of $\mathcal{C}$, and $1_{\operatorname{dom} w} = (w^{-1}, w)$ and $1_{\operatorname{codom} w} = (w, w^{-1})$ for all $w \in \mathcal{W}$. Here $\operatorname{dom} w$ is the domain of w and $\operatorname{codom} w$ is the codomain of w.

Definition 6.8. Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories.

(1) If $F : \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, define the total left derived functor $LF : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$ to be the composite

$$\mathrm{Ho}\mathcal{C} \xrightarrow{\mathrm{Ho}Q} \mathrm{Ho}\mathcal{C}_c \xrightarrow{\mathrm{Ho}F} \mathrm{Ho}\mathcal{D},$$

Given a natural transformation $\tau : F \to F'$ of left Quillen functors, define the total derived natural transformation $L\tau$ to be $\mathrm{Ho}\tau \circ \mathrm{Ho}Q$, so that $(L\tau)_X = \tau_{QX}$.

(2) If $U : \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, define the total right derived functor $RU : \mathrm{Ho}\mathcal{D} \to \mathrm{Ho}\mathcal{C}$ to be the composite

$$\mathrm{Ho}\mathcal{D} \xrightarrow{\mathrm{Ho}R} \mathrm{Ho}\mathcal{D}_f \xrightarrow{\mathrm{Ho}U} \mathrm{Ho}\mathcal{C},$$

Given a natural transformation $\tau : U \to U'$ of right Quillen functors, define the total derived natural transformation $R\tau$ to be $\mathrm{Ho}\tau \circ \mathrm{Ho}R$, so that $(R\tau)_X = \tau_{RX}$.

Definition 6.9 (Quillen equivalence)**.** A Quillen adjunction is called a Quillen equivalence if its left and right derived functors are equivalences of homotopy categories.

Finally, we can state the theorem which gives us the equivalence of our constructions. For the proof and the relevant model structures on the categories, the reader is advised to refer to [3].

Theorem 6.10 (Quillen)**.** *The adjunction*

$$|-|: \mathrm{Set}_\Delta \rightleftarrows \mathcal{CG}: \mathrm{Sing}$$

is a Quillen equivalence of the Kan-Quillen model structure on Set_Δ *and the classical model structure on* $\mathcal{CG}$*.*

7. The ∞-category of spaces

Now that we have two equivalent definitions of ∞-categories, we can better understand the underlying structure. The first step in this direction is considering a generalization of Set for ∞-categories. In 1-category theory, with our axiom, all categories are enriched over Set. Even in constructions without the axiom, our categories are often taken to be locally small or, equivalently, enriched over Set. In this section, we aim to develop an ∞-category $\mathcal{S}$ such that every ∞-category is enriched over $\mathcal{S}$.

Note that every $(\infty, 0)$-category is a Kan complex because we can extend maps from the horn to the standard n-simplex by simply using weak inverses of maps. For example, given an outer horn diagram

$$\begin{tikzcd} & X_2 & \\ X_1 \arrow[ur, "f_{12}"] & & X_0 \arrow[ul, "f_{02}"'] \end{tikzcd}$$

We can find a dotted arrow $g = f_{12}^{-1} \circ f_{02}$ where f_{12}^{-1} is the homotopy inverse of f_{02} and a homotopy equivalence σ such that the following diagram commutes.

$$\begin{tikzcd} & X_2 \arrow[d, phantom, "\Downarrow\sigma"] & \\ X_1 \arrow[ur, "f_{12}"] & & X_0 \arrow[ul, "f_{02}"'] \arrow[ll, dashed, "g=f_{12}^{-1}\circ f_{02}"] \end{tikzcd}$$

For the enrichment, we first define the category of Kan complexes.

Definition 7.1 ($\mathcal{K}$an)**.** We denote by $\mathcal{K}$an the full subcategory of Set_Δ given by the collection of Kan complexes.

Then we may naively assert that every ∞-category is enriched over $\mathcal{K}$an, but this is not what we want, for $\mathcal{K}$an is not even an ∞-category. Still, we could use the canonical way of attaining an ∞-category from an ordinary category: applying the nerve functor. Unfortunately, our previous construction of a nerve functor will be inadequate in this case. $\mathcal{K}$an can be easily shown to be a category enriched over Set_Δ, but the nerve functor forgets the existing simplicial data in a category and builds one from the ground up using composable strings. So, first, we will introduce an improvement of our previous nerve functor, called the homotopy coherent nerve, as presented in [9]. To capture the simplicial structure, we start with a "thickening" of the category $[n]$, which we will denote by $\mathfrak{C}[\Delta^n]$.

Definition 7.2. Let J be a finite nonempty linearly ordered set. The simplicial category $\mathfrak{C}[\Delta^J]$ is defined as follows.

- The objects of $\mathfrak{C}[\Delta^J]$ are the elements of J
- If $i, j \in J$ then

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) = \begin{cases} \varnothing & \text{if } j < i, \\ \mathrm{N}(P_{i,j}) & \text{if } i \leq j. \end{cases}$$

  where $P_{i,j}$ is the poset $\{I \subset J \mid i, j \in I \text{ and } i \leq k \leq j \text{ for all } k \in I\}$
- If $i_0 \leq i_1 \leq \cdots \leq i_n$, then the composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0,i_1) \times \cdots \times \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_{n-1},i_n) \to \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0,i_n)$$

  is induced by the map of posets

$$P_{i_0,i_1} \times \cdots \times P_{i_{n-1},i_n} \to P_{i_0,i_n}$$
$$(I_1, \cdots, I_n) \mapsto I_1 \cup \cdots \cup I_n$$

We advise the interested reader to refer to Remark 1.1.5.2 of [9] to better understand the relationship between $[n]$ and $\mathfrak{C}[\Delta^n]$.

Definition 7.3 (Homotopy coherent nerve)**.** Let $\mathcal{C}\mathrm{at}_\Delta$ denote the category of categories enriched over Set_Δ. The homotopy coherent nerve, also called the simplicial nerve, $\mathrm{N}_\Delta(\mathcal{C})$ of a simplicial category $\mathcal{C}$ is given by

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{N}_\Delta(\mathcal{C})) = \mathrm{Hom}_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$$

Definition 7.4 (S)**.** We call $N_\Delta(\mathcal{K}\mathrm{an})$ the ∞-category of spaces and denote it by $\mathcal{S}$.

Now, the below proposition easily follows.

Proposition 7.5. Every ∞-category is enriched over $\mathcal{S}$.

Remark 7.6. There are other ways to define $\mathcal{S}$, such as the topological nerve (defined as $N_\Delta(\mathrm{Sing}\,\mathcal{C})$ for a topological category $\mathcal{C}$) of the category of CW complexes and continuous maps, which is equivalent to our construction. In the end, Definition 7.4 was chosen to make the proof of ∞-categorical Yoneda lemma in [9] easier for the willing reader.

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